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Dynamics of Rotating Flexible Structures by a Method of Quadratic Modes

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Dynamics of Rotating Flexible Structures by a Method of Quadratic Modes

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Abstract

The problem of calculating the vibrations of rotating structures has challenged analysts since the observation that use of traditional modal coordinates in such problems leads to the prediction of instability involving infinite deformation when rotation rates exceed the first natural frequency. Much recent published work on beams has shown that such predictions are artifacts of incorporating incomplete kinematics into the analysis, but that work addresses analysis of only simple structures such as individual beams and plates. The authors present a new approach to analyzing rotating flexible structures that applies to the rotation of *general* linear (unjointed) structures, using a system of nonlinearly coupled deformation modes. This technique is called a *Method of Quadratic Modes.*

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1 Introduction

The literature discussing the difficulties of calculating the vibration of rotating structures is extensive (Ref. [1] through [29]). An illuminating comparison of various approaches is given by Ryan and Yoo [19]. Much of that discussion focuses on rotating beams and on the inadequacy of the standard, simple extensions to multibody codes designed for rigid structures to predict the behavior of flexible bodies. Those difficulties have been identified as having to do with kinematics that are of only second order importance in non-rotating problems, but that become of first order importance in problems with significant angular velocities. These additional kinematics are also important where there are external force fields present. These kinematics are usually "lost" in normal linearization processes.

These "lost kinematics" are most easily illuminated in the case of a flexible but inextensible beam fixed at one end and vibrating in a plane (See Figure 1). The beam is assumed to be inextensible, so its lateral vibration must be accompanied by longitudinal inward displacements. For small vibrations, if the lateral motions are of magnitude s, these "foreshortening" displacements are of magnitude s^2, and their associated kinetic energies are of fourth order and don't interfere with the lateral motions in ordinary vibration problems. However, if there is a preexisting tension in the beam, e.g. due to an external field such as gravity, the work associated with the foreshortening becomes of order s^2 and becomes an important factor in the coupled lateral vibration.

If the above beam were rotated about its hub, the longitudinal tension would cause

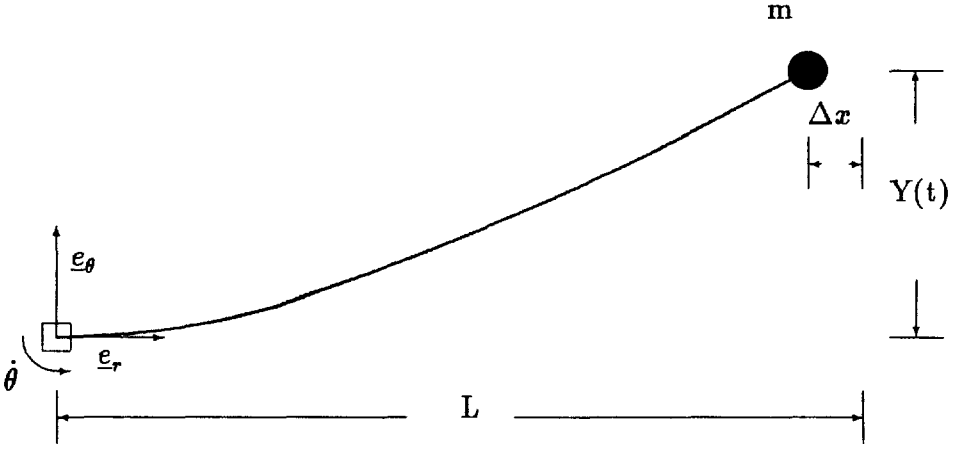

Figure 1. A deformed, rotating beam. The undeformed beam length is L, and the current lateral deformation is Y(t).

centrifugal stiffening of lateral vibrations. Though the complicated vibrations of the beam can be approximated by adding *a posteriori* forces to the mechanics of a stationary beam, this approach does not generate the kind of rigorous certainty that is expected in modern mechanics and is necessary for quantitative prediction.

The equations for this example are developed below, where for simplicity, the beam is taken to be massless but supporting a mass m at its end. We consider the hub to be rotating with a specified angular velocity $\dot{\theta}(t)$ and solve for the lateral displacement of the tip mass $Y(t)$ relative to a frame rotating with the hub. Since the beam is *massless*, the lateral displacement of points on the beam between the hub and the end mass can be expressed as the product of the static displacement shape and the displacement of the end mass:

$$y(rL,t) = f(r)Y(t) \tag{1}$$

where L is the original length of the beam, r ranges from 0 to 1, and f is a function characteristic of the stiffness properties of the beam [1] and normalized so that $f(1) = 1$. The axial displacement corresponding to that lateral motion is

$$\Delta x(t) = -\alpha Y(t)^2/L \tag{2}$$

where

$$\alpha = \int_0^1 \frac{1}{2}(\frac{df}{dr})^2 dr. \tag{3}$$

(Derivation of the above expression for α employs the binomial expansion for $\left[1 - (\frac{Y(t)}{L}\frac{df}{dr})^2\right]^{1/2}$).

A direct application of the calculus of variations shows that the right hand side of Equation 3 for α is minimized when $f(r) = r$ and that for all $f(r)$, [2]

$$\alpha \geq \frac{1}{2}. \tag{4}$$

The position of the end-mass is:

$$\underline{x}(t) = \left[L - \alpha Y(t)^2/L\right] \underline{e}_r(t) + Y(t)\underline{e}_\theta(t), \tag{5}$$

where $\underline{e}_r(t)$ is a unit vector tangent to the beam at the hub and $\underline{e}_\theta(t)$ is a unit vector also rotating with the hub, but oriented 90° counterclockwise from $\underline{e}_r(t)$.

The velocity of the end-mass is:

$$\begin{aligned} \dot{\underline{x}}(t) = \quad & -Y(t)\{\,\dot{\theta}(t) + 2\alpha\left[\dot{Y}(t)/L\right]\}\underline{e}_r(t) \\ & +\{\,\dot{\theta}(t)\left[L - \alpha Y(t)^2/L\right] + \dot{Y}(t)\}\underline{e}_\theta(t) \end{aligned} \tag{6}$$

[1] For a uniform cross section Euler-Bernoulli beam, $f(r) = (3/2)r^2(1 - r/3)$.

[2] For an Euler-Bernoulli beam of uniform cross section, $\alpha = \frac{3}{5}$.

In deriving the above equation, use has been made of the observations that

$$\dot{\underline{e}}_r = \dot{\theta}\,\underline{e}_\theta \tag{7}$$

and

$$\dot{\underline{e}}_\theta = -\dot{\theta}\,\underline{e}_r \tag{8}$$

Retaining at most quadratic terms in $Y(t)$ and $\dot{Y}(t)$, the kinetic energy is:

$$KE = \frac{1}{2}m\{\dot{\theta}(t)^2\left[L^2 + Y(t)^2(1-2\alpha)\right] + 2\dot{Y}(t)\dot{\theta}(t)L + \dot{Y}(t)^2\} \tag{9}$$

Expressing the spring stiffness of the beam [3] as κ, the strain energy is:

$$PE = \frac{1}{2}\kappa Y(t)^2 \tag{10}$$

Using these expressions for KE and PE in Lagrange's equation, the following governing equation of motion is derived:

$$m\ddot{Y}(t) + \left[m\dot{\theta}(t)^2(2\alpha - 1) + \kappa\right] Y(t) = -mL\ddot{\theta}(t) \tag{11}$$

Since $\alpha > \frac{1}{2}$, the term within the brackets in the above equation is always positive and for any set of initial conditions, there exist bounded solutions to the differential equation. Had the inward kinematics not been considered, the terms involving α would not have been incorporated in the above equation and the solutions would have been unbounded for rotation rates $\|\dot{\theta}\| > \sqrt{\kappa/m}$. These non-physical, unbounded solutions are sometimes sarcastically referred to as "buckling in tension".

An important observation is that though foreshortening was introduced in the above example problem through second order terms in $Y(t)$, their impact is manifest through a term in Equation 11 that is *linear* in $Y(t)$: the resulting linear equation in Y contains terms resulting from the nonlinear kinematics. A conclusion of the above calculation is that the displacement field must be correct up to at least the second order in the primary deformation [8, 9].

Several approaches have been investigated recently for incorporating correct dynamics into simple problems such as this. These methods include:

- Introducing a higher order strain measure in the evaluation of strain energy. The strain measures employed in this approach are those that were devised in classical finite elasticity to be insensitive to rigid body rotation[30, 31]. The higher order

[3] For an Euler-Bernoulli beam of uniform cross section, $\kappa = 3EI/L^3$.

strain may be used to evaluate strain energy associated with extension and that strain energy is used in deriving the governing equations for the beam.

Linearization about the deformed kinematics results in equations which generate the expected "stiffening". This is the approach used by Simo and Vu-Quoc[10]. Reference [10] presents a very clear discussion of the nonlinearities associated with the rotating beam and the rotating plate.

Detailed developments using finite strain measures are presented elsewhere by Simo and Vu-Quoc[11, 12].

- A similar high order strain is used to evaluate the stretch along the beam as it curves. Incorporating the axial strain associated with the axial stretch as a field in the governing equations results in deformed shapes having an appropriate "foreshortening". This is, in effect, the approach used by Kane, Ryan, and Banerjee [13].

- Likins et. al [1] found steady state solutions about which to linearize oscillatory solutions. They used a nonlinear strain measure to impose the appropriate kinematic constraints and employed the resulting membrane stresses as a preload on the vibrations problem. Such an approach had earlier been pursued by Craig[2] to study the vibrations of a rotating beam. A similar approach was employed by Carne et. al. [3] (later republished as [4]) and later employed by Lobitz [5, 6] in the calculation of the vibrations of wind turbines. They used the nonlinear capabilities of NASTRAN, including the generation of geometric stiffness matrices, to achieve both the steady state configuration and the linearized stiffness matrix.

- Laurenson[14] introduced geometric stiffness matrices to account for the effect of the "preload" due to centrifugal accelerations. Lawrenson[15] later made a more rigorous presentation of that approach. Zeiler and Buttrill [16] used geometric stiffness matrices in connection with a finite element code to account for transient angular velocity. Idler and Amirouche [17] developed equations of motion for coupled flexible beam-like structures, using a geometric stiffness matrix to account for the coupling between inertial terms and transverse deformations of the substructures. Banerjee and Dickens [18] recently implemented that approach again in the context of finite elements.

- A nonlinear finite element routine accommodating the nonlinearities of the problem through recalculation of mass and stiffness matrices at each time step may be used. An example of such a calculation is that of Peterson [21] using a commercial finite element code. Christensen and Lee [22] used a nonlinear strain measure to derive a finite element formulation which required recalculaton of the stiffness matrix at each time step. In this same spirit, Wu and Haug [23] employ a system of substructures, the deformations of each substructure being defined with respect to its own locally rotating frame.

The literature on this topic is extensive and is outlined in references [24] and [25].

It is now clear that if linear techniques (such as modal superposition) can be used in this intrinsically nonlinear problem, they can not be the standard techniques of linear analysis. This is established by the observation that in general, the relevant nonlinear kinematics can not be captured by combinations of the linear modes. Yet, a system of generalized degrees of freedom is especially important in nonlinear problems since nonlinear solutions are intrinsically expensive and reducing the degrees of freedom reduces that expense. Further, assembling the full nonlinear systems of equations for a complete finite element discretization at every time step is generally prohibitive.

It should also be noted that the above example problem was solved making use of prior knowledge of exactly which constraint was active and of the nonlinear kinematics necessary to satisfy that constraint. What is needed is a technique that will automatically identify the relevant constraints and the kinematics necessary to satisfy those constraints.

2 A Method of Quadratic Modes

What is required is a technique employing a small number of generalized equations that adequately account for the geometric nonlinearities of these problems and does not require the recalculation of stiffness matrices at every time step. **It is also important that such a method address general three-dimensional, linear structures, not just simple beams or plates.**

Such a technique is presented here. This approach is motivated by the need for a rigorous (predictive) method to calculate the vibratory response of general structures, and in a way that does not require *a priori* knowledge of the "lost kinematics". The essence of the approach is to use information about the nonlinear static response of the structure to define a space of configurations which, to second order, are consistent with kinematic constraints such as those associated with "foreshortening". (Those kinematics won't necessarily appear as "foreshortening" in more complex structures.) These kinematics are used with Hamilton's principle to arrive at a relatively small system of governing equations for the dynamics of the rotating, vibrating structure.

The necessary kinematics are introduced as functions of a basis of applied force fields. The displacement field is expressed as a nonlinear operator of the applied force field:

$$\mathcal{U} = N(\mathcal{F}) \tag{12}$$

where $\mathcal{U}$ is the displacement field resulting from the force field $\mathcal{F}$ and N is the nonlinear mapping from $\mathcal{F}$ to $\mathcal{U}$. The force field $\mathcal{F}$ consists of point forces, distributed surface tractions, and distributed body forces.

The nonlinear operator N is expanded in a Taylor series which is truncated after

the quadratic term:

$$\mathcal{U} = L(\mathcal{F}) + B(\mathcal{F}, \mathcal{F}) \tag{13}$$

where L is a linear operator and B is a bilinear operator. These operators are defined in terms of Frechét derivatives of the nonlinear operator N [32]. Numerical methods for evaluating these operators is presented in a later section.

Next, a basis of force fields $\{\mathcal{F}^i\}$ is considered:

$$\mathcal{F} = s_i \mathcal{F}^i \tag{14}$$

where each field $\mathcal{F}^i$ is time independent and where summation is performed on repeated indices, as it is throughout the remainder of this paper. These serve as generators of the nonlinear space of displacement configurations.

Substituting Equation 14 into Equation 13, the static displacement field is now:

$$\mathcal{U}(\{s_i\}) = s_i \mathcal{U}^i + s_i s_j \mathcal{G}^{ij} \tag{15}$$

where

$$\mathcal{U}^i = L(\mathcal{F}^i) \tag{16}$$

and

$$\mathcal{G}^{ij} = B(\mathcal{F}^i, \mathcal{F}^j) \tag{17}$$

Since $\mathcal{G}^{ij}$ is the second Frechét derivative of $\mathcal{U}$ with respect to each of $\mathcal{F}^i$ and $\mathcal{F}^j$, $\mathcal{G}^{ij} = \mathcal{G}^{ji}$.

The above displacement fields are evaluated at individual particles $\underline{\chi}$ to yield:

$$\underline{u}(\underline{\chi}, t) = s_i(t) \underline{u}^i(\underline{\chi}) + s_i(t) s_j(t) \underline{g}^{ij}(\underline{\chi}) \tag{18}$$

The fields $\mathcal{U}^i$ and $\mathcal{G}^{ij}$ are constant, as are their values at individual points, $\underline{u}^i(\underline{\chi})$ and $\underline{g}^{ij}(\underline{\chi})$. The symmetry of $\mathcal{G}^{ij}$ in its indices carries over to the symmetry of g^{ij} in its indices. For convenience, the above will sometimes be represented by the notation

$$\underline{u}(\underline{\chi}, t) = \underline{U}(\{s_i(t)\}, \underline{\chi}) \tag{19}$$

Similarly, the imposed force fields can be evaluated at individual particles:

$$\underline{f}(\underline{\chi}, t) = s_i(t)\, \underline{f}^i(\underline{\chi}) \tag{20}$$

The displacement components $\mathcal{U}^i$ and $\mathcal{G}^{ij}$ can be determined through limiting processes involving N:

$$\mathcal{U}^i = \frac{N(s_i \mathcal{F}^i) - N(-s_i \mathcal{F}^i)}{2 s_i} + O(s_i^2); \tag{21}$$

$$\mathcal{G}^{ii} = \frac{N(s_i \mathcal{F}^i) + N(-s_i \mathcal{F}^i)}{2 s_i^2} + O(s_i^2); \tag{22}$$

(with no sum on the i), and

$$\mathcal{G}^{ij} = [N(s_i\mathcal{F}^i + s_j\mathcal{F}^j) + N(-s_i\mathcal{F}^i - s_j\mathcal{F}^j) - N(s_i\mathcal{F}^i) - N(s_j\mathcal{F}^j) - N(-s_i\mathcal{F}^i) - N(-s_j\mathcal{F}^j)]/4s_is_j + O((|s_i| + |s_j|)^2) \tag{23}$$

with no sum on the i or j) for $i \neq j$.

Greater precision is achieved by evaluating N at more points

$$\mathcal{U}^i = \frac{8\,[N(s_i\mathcal{F}^i) - N(-s_i\mathcal{F}^i)] - [N(2s_i\mathcal{F}^i) - N(-2s_i\mathcal{F}^i)]}{12s_i} + O(s_i^4) \tag{24}$$

and

$$\mathcal{G}^{ii} = \{16\,[N(s_i\mathcal{F}^i) + N(-s_i\mathcal{F}^i)] - [N(2s_i\mathcal{F}^i) + N(-2s_i\mathcal{F}^i)]\}/24s_i^2 + O(s_i^4) \tag{25}$$

(with no sum on the i), and

$$\mathcal{G}^{ij} = \{16[\mathcal{N}(s_i\mathcal{F}^i + s_j\mathcal{F}^j) + \mathcal{N}(-s_i\mathcal{F}^i - s_j\mathcal{F}^j) - \mathcal{N}(s_i\mathcal{F}^i) - \mathcal{N}(-s_i\mathcal{F}^i) - \mathcal{N}(s_j\mathcal{F}^j) - \mathcal{N}(-s_j\mathcal{F}^j)] - [\mathcal{N}(2s_i\mathcal{F}^i + 2s_j\mathcal{F}^j) + \mathcal{N}(-2s_i\mathcal{F}^i - 2s_j\mathcal{F}^j) - \mathcal{N}(2s_i\mathcal{F}^i) - \mathcal{N}(-2s_i\mathcal{F}^i) - \mathcal{N}(2s_j\mathcal{F}^j) - \mathcal{N}(-2s_j\mathcal{F}^j)]\}/48s_is_j + O((|s_i| + |s_j|)^4) \tag{26}$$

(with no sum on the i or j) for $j \neq i$.

The above six equations are derived by truncating the Taylor series, evaluating that truncated series at appropriate points, and formally solving for the coefficients of the linear and quadratic terms. This process, which is equivalent to standard methods of numerical differentiation [33], could be continued to achieve expressions for $\mathcal{U}^i$ and $\mathcal{G}^{ij}$ of arbitrary order in s_i.

It is important to observe here that for all sets of coefficients $\{s_i\}$, **the resulting displacement field does satisfy all kinematic constraints up to second order.** It is the $\{s_i\}$ which are next used as generalized degrees of freedom.

Appropriate bases of forces $\mathcal{F}^i$ may be selected to reflect either static or modal loadings. The elastic modal deformations, along with resulting quadratic terms, are introduced in the following manner:

1. Let $\mathcal{Y}^n$ be the n'th (linear) eigenmode, satisfying $-\rho(\underline{x})\omega_n^2\mathcal{Y}^n(\underline{x}) + D_{\underline{x}}\mathcal{Y}^n(\underline{x}) = 0$ where there is no sum on n and $D_{\underline{x}}$ is the associated spatial operator.

2. Define $\mathcal{F}^n = D_{\underline{x}}\mathcal{Y}^n$

Static modes are introduced simply by defining the $\mathcal{F}^n$ to be the externally imposed loads associated with those static modes.

Completeness of the basis force fields $\mathcal{F}^i$ is assured by requiring that the resulting linear displacement fields $\mathcal{U}^i$ are complete in the corresponding linear space.

3 Governing Equations

3.1 Kinematics of a Flexible Body

The position of a particle on a flexible body undergoing large motions is represented as the result of the successive application of deformation, rigid-body rotation, and rigid body translation (See Figure 2):

$$\underline{x}(\underline{\chi}, t) = \underline{p}(t) + \mathbf{R}(t) \cdot [\underline{\chi} + \underline{u}(\underline{\chi}, t)] \tag{27}$$

where $\underline{\chi}$ is the reference location of the given particle; $\underline{u}(\underline{\chi}, t)$ is the displacement from the reference location at time t due to deformation $\underline{u}$; $\mathbf{R}$ is the rotation from the reference configuration to the configuration at time t; and $\underline{p}(t)$ is the translation from the reference configuration to the configuration at time t. In component form, the above equation is:

$$x_i(\underline{\chi}, t) = p_i(t) + R_{ij}(t) \left[\chi_j + u_j(\underline{\chi}, t)\right] . \tag{28}$$

Note that $\mathbf{R}(t)$ and $\underline{p}(t)$ are rotation and translation as seen in an inertial reference frame, so that $\dot{\underline{x}}$ will be velocity as seen in that inertial reference frame.

In a frame that translates with $\underline{p}(t)$ and rotates along with $\mathbf{R}(t)$, the only displacements are $\underline{u}(\underline{\chi}, t)$. Such "floating" frames have been discussed in detail in the literature [26, 27, 28]. In particular Reference [29] contains a recent discussion on the relative advantages of defining $\underline{p}$ and $\mathbf{R}$ to be either the translation of the center of mass and the rotation of the principle axes of inertia, or the translation and rotation of a specific particle on the body.

The development of the equations of motion requires the following easily-verified kinematic relations:

$$\frac{d\mathbf{R}}{dt} = \mathbf{\Omega} \cdot \mathbf{R}, \tag{29}$$

and

$$\delta\mathbf{R} = \mathbf{\Lambda} \cdot \mathbf{R} \tag{30}$$

where $\mathbf{\Omega}$ is the spin tensor corresponding to the angular velocity $\underline{\omega}$ at the current time, and $\mathbf{\Lambda}$ is the tensor of virtual rotations about the current configurations.

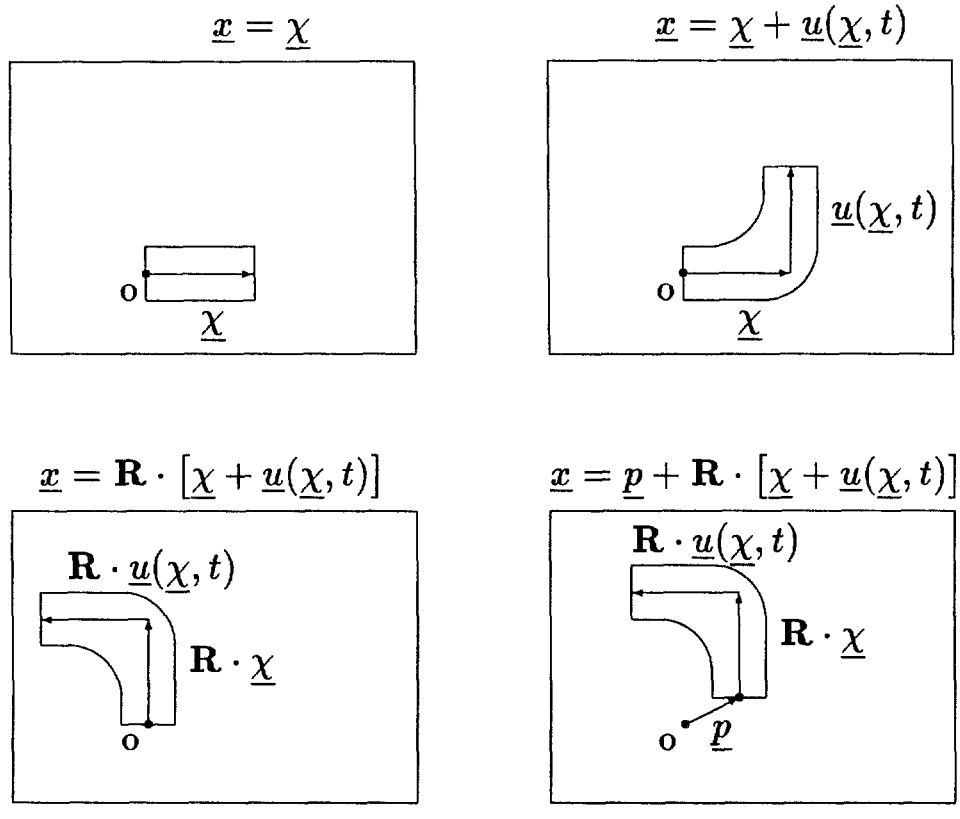

Figure 2. The configuration at time t is the net result of deformation, rotation, and translation.

The spin tensor is related to the angular velocity vector through the alternation tensor ϵ:

$$\omega_i = -\frac{1}{2}\epsilon_{ijk}\mathbf{\Omega}_{jk}\,, \tag{31}$$

and the tensor of virtual rotations is similarly related to a corresponding vector of virtual rotations:

$$\lambda_i = -\frac{1}{2}\epsilon_{ijk}\mathbf{\Lambda}_{jk}\,. \tag{32}$$

Equations 29 and 30 for the time derivative and variation of $\mathbf{R}$ can be restated now in terms of vector/tensor operators:

$$\frac{d\mathbf{R}}{dt} = \underline{\omega} \times \mathbf{R}\,, \tag{33}$$

and

$$\delta\mathbf{R} = \underline{\lambda} \times \mathbf{R}\,. \tag{34}$$

Using the above relations, one obtains the following expressions for velocity, acceleration, and virtual displacement of the particle originally located at $\underline{\chi}$:

$$\dot{\underline{x}}(\underline{\chi},t) = \dot{\underline{p}} + \mathbf{\Omega}\cdot\mathbf{R}\cdot\left[\underline{\chi} + \underline{u}(\underline{\chi},t)\right] + \mathbf{R}\cdot\dot{\underline{u}}(\underline{\chi},t)\,; \tag{35}$$

$$\ddot{\underline{x}}(\underline{\chi},t) = \ddot{\underline{p}} + (\dot{\mathbf{\Omega}} + \mathbf{\Omega}\cdot\mathbf{\Omega})\cdot\mathbf{R}\cdot\left[\underline{\chi} + \underline{u}(\underline{\chi},t)\right] + 2\mathbf{\Omega}\cdot\mathbf{R}\cdot\dot{\underline{u}} + \mathbf{R}\cdot\ddot{\underline{u}}(\underline{\chi},t)\,; \tag{36}$$

and

$$\delta\underline{x}(\underline{\chi},t) = \delta\underline{p} + \underline{\lambda}\times\left\{\mathbf{R}\cdot\left[\underline{\chi} + \underline{u}(\underline{\chi},t)\right]\right\} + \mathbf{R}\cdot\delta\underline{u}(\underline{\chi},t)\,. \tag{37}$$

The above expressions will be made slightly longer with the employment of the earlier expression for $\underline{u}$:

$$\underline{u}(\underline{\chi},t) = \underline{U}(\{s_i(t)\},\underline{\chi}) \tag{38}$$

so $\delta\underline{u}$, $\dot{\underline{u}}$, and $\ddot{\underline{u}}$ have simple forms:

$$\delta\underline{u} = \frac{\partial\underline{U}}{\partial s_i}\delta s_i = \delta s_i\underline{u}^i + 2\delta s_i s_j\underline{g}^{ij}\,; \tag{39}$$

$$\dot{\underline{u}} = \frac{\partial\underline{U}}{\partial s_i}\dot{s}_i = \dot{s}_i\underline{u}^i + 2\dot{s}_i s_j\underline{g}^{ij}\,; \tag{40}$$

and

$$\ddot{\underline{u}} = \frac{\partial^2\underline{U}}{\partial s_i\partial s_j}\dot{s}_i\dot{s}_j + \frac{\partial\underline{U}}{\partial s_i}\ddot{s}_i = \ddot{s}_i(\underline{u}^i + 2s_j\underline{g}^{ij}) + 2\dot{s}_i\dot{s}_j\underline{g}^{ij}\,. \tag{41}$$

3.2 Hamilton's Principle

The extended Hamilton's principle [34] asserts that the first variation of the time integral of the Lagrangian of a mechanical system plus the variation of the work of external forces is made stationary by the equations of motion of that system. In the context of this *elastic* system, and ignoring initial and terminal conditions:

$$0 = \int_{t_1}^{t_2} [\delta KE - \delta SE + \delta XE]\, dt \tag{42}$$

where KE is the kinetic energy; SE is strain energy; and δXE the virtual work of external forces. Hamilton's principle asserts that the integral on the right hand side of the above equation must be stationary with respect to all independent kinematic variables.

The kinetic energy is

$$KE = \frac{1}{2} \int_{vol} \rho(\underline{\chi})\, \dot{\underline{x}}(\underline{\chi}, t) \cdot \dot{\underline{x}}(\underline{\chi}, t)\, dV(\underline{\chi}) \tag{43}$$

The variation of kinetic energy in Hamilton's principle is:

$$\int_{t_1}^{t_2} \delta KE\, dt = \int_{t_1}^{t_2} \int_{vol} \rho(\underline{\chi})\, \delta\dot{\underline{x}}(\underline{\chi}, t) \cdot \dot{\underline{x}}(\underline{\chi}, t)\, dV(\underline{\chi})\, dt \tag{44}$$

On integration by parts to resolve out variations in velocity for variations in displacement, this becomes:

$$\int_{t_1}^{t_2} \delta KE\, dt = \int_{t_1}^{t_2} \int_{vol} -\rho(\underline{\chi})\, \delta\underline{x}(\underline{\chi}, t) \cdot \ddot{\underline{x}}(\underline{\chi}, t)\, dV(\underline{\chi}) dt \tag{45}$$

This integration by parts may also be performed through the more laborious process of first expanding $\delta\ddot{\underline{x}}$ and $\delta\underline{x}$ as in Equations 35 and 37 and performing integrations-by-parts as necessary to remove variations in rate terms. This second approach requires use of the lemma proven in Appendix I.

The strain energy is:

$$SE = \frac{1}{2} Q(\mathcal{U}, \mathcal{U}) \tag{46}$$

where $\mathcal{U}$ is the whole field of displacements $\underline{u}(\underline{\chi}, t)$ with respect to the initial configuration and Q is a quadratic operator characteristic of the structure. The variation in strain energy is:

$$\int_{t_1}^{t_2} \delta SE\, dt = \int_{t_1}^{t_2} Q(\delta\mathcal{U}, \mathcal{U})\, dt \tag{47}$$

The above is a linear operator on $\mathcal{U}$ and may be written as an inner product:[4]

$$\int_{t_1}^{t_2} \delta SE \, dt = \int_{t_1}^{t_2} \int_{vol} \delta \underline{u}(\underline{\chi}, t) \cdot \underline{F_s(\mathcal{U})}(\underline{\chi}, t) \, dV(\underline{\chi}) \, dt \tag{48}$$

where $\underline{F_s(\mathcal{U})}(\underline{\chi}, t)$ is the distributed force field in the vicinity of $\underline{\chi}$ associated with the static displacement field $\mathcal{U}$; $\underline{F_s(\mathcal{U})}(\underline{\chi}, t)$ is the field of reaction forces due to $\mathcal{U}$. To within the linear response of the structure,

$$\underline{F_s(\mathcal{U})}(\underline{\chi}, t) = s_i(t) \, \underline{f}^i(\underline{\chi}) \tag{49}$$

The variation in external work is:

$$\int_{t_1}^{t_2} \delta XE \, dt = \int_{t_1}^{t_2} \int_{vol} \delta \underline{x}(\underline{\chi}, t) \cdot \underline{F_X}(\underline{\chi}, t) \, dV(\underline{\chi}) \, dt \tag{50}$$

where $\delta \underline{x}(\underline{\chi}, t)$ is the virtual displacement with respect to the inertial frame and $\underline{F_X}(\underline{\chi}, t)$ is the distributed externally applied force field in the vicinity of $\underline{\chi}$.

Combining the constituent terms into the integral of Hamilton's principle, we obtain:

$$\begin{aligned} 0 = \int_{t_1}^{t_2} \int_{vol} \{ & \delta \underline{x}(\underline{\chi}, t) \cdot \left[-\rho(\underline{\chi}) \, \ddot{\underline{x}}(\underline{\chi}, t) + \underline{F_X}(\underline{\chi}, t) \right] \\ & - \delta \underline{u}(\underline{\chi}, t) \cdot \underline{F_s(\mathcal{U})}(\underline{\chi}, t) \} \, dV(\underline{\chi}) \, dt \end{aligned} \tag{51}$$

The above integral must hold for all $\delta \underline{x}(\underline{\chi}, t)$, generating the following conditions, one for each component of $\delta \underline{x}$, that must hold at all times $t_1 \leq t \leq t_2$:

$$0 = \int_{vol} \delta \underline{p} \cdot \left[-\rho(\underline{\chi}) \, \ddot{\underline{x}}(\underline{\chi}, t) + \underline{F_X}(\underline{\chi}, t) \right] \, dV(\underline{\chi}) \tag{52}$$

$$0 = \int_{vol} (\underline{\lambda} \times \{ \mathbf{R} \cdot [\underline{\chi} + \underline{u}(\underline{\chi}, t)] \}) \cdot \left[-\rho(\underline{\chi}) \, \ddot{\underline{x}}(\underline{\chi}, t) + \underline{F_X}(\underline{\chi}, t) \right] \, dV(\underline{\chi}) \tag{53}$$

$$0 = \delta s_i \int_{vol} (\mathbf{R} \cdot \frac{\partial U}{\partial s_i}) \cdot \left[-\rho(\underline{\chi}) \ddot{\underline{x}} + \underline{F_X}(\underline{\chi}, t) - \underline{F_s(\mathcal{U})}(\underline{\chi}, t) \right] \, dV(\underline{\chi}) \tag{54}$$

Since $\delta \underline{p}$, $\underline{\lambda}$, and δs_i are all arbitrary, their coefficients must each be zero. Note that there are exactly as many equations as there are acceleration terms: three scalar equations associated with $\underline{p}$ for the three components of $\ddot{\underline{p}}$; three scalar equations associated with $\underline{\lambda}$ for the three independent components of $\dot{\mathbf{\Omega}}$; and one equation associated with each δs_i for the corresponding $\ddot{s}_i$.

[4]The justification for this representation lies with the famous theorem of F. Reisz. A simple discussion is found in Reference [36].

The above equations may be mapped from the inertial frame to a frame coincident with the original configuration. This new frame may be seen as a co-rotating frame. Letting:

$$\hat{\delta p} = \mathbf{R}^T \cdot \delta \underline{p} \tag{55}$$

$$\hat{\ddot{p}} = \mathbf{R}^T \cdot \ddot{p} \tag{56}$$

$$\hat{\underline{\lambda}} = \mathbf{R}^T \cdot \underline{\lambda} \tag{57}$$

$$\hat{\ddot{\underline{x}}} = \mathbf{R}^T \cdot \ddot{\underline{x}} \tag{58}$$

$$\underline{\hat{F}_X} = \mathbf{R}^T \cdot \underline{F_X} \tag{59}$$

$$\underline{\hat{F}_s} = \mathbf{R}^T \cdot \underline{F_s} \tag{60}$$

$$\hat{\Omega} = \mathbf{R}^T \cdot \Omega \cdot \mathbf{R} \tag{61}$$

$$\hat{\dot{\Omega}} = \mathbf{R}^T \cdot \dot{\Omega} \cdot \mathbf{R} \tag{62}$$

the equations of motion for the body become:

$$0 = \delta \hat{\underline{p}} \cdot \int_{vol} \left[-\rho(\underline{\chi}) \hat{\ddot{\underline{x}}}(\underline{\chi}, t) + \underline{\hat{F}_X}(\underline{\chi}, t) \right] dV(\underline{\chi}) \tag{63}$$

$$0 = \int_{vol} (\hat{\underline{\lambda}} \times [\underline{\chi} + s_i \underline{u}^i(\underline{\chi})]) \cdot \left[-\rho(\underline{\chi}) \hat{\ddot{\underline{x}}}(\underline{\chi}, t) + \underline{\hat{F}_X}(\underline{\chi}, t) \right] dV(\underline{\chi}) \tag{64}$$

$$0 = \delta s_i \int_{vol} [\underline{u}^i(\underline{\chi}) + 2 s_j \underline{g}^{ij}(\underline{\chi})] \cdot \left[-\rho(\underline{\chi}) \hat{\ddot{\underline{x}}} + \underline{\hat{F}_X}(\underline{\chi}, t) - \underline{\hat{F}_s(\mathcal{U})}(\underline{\chi}, t) \right] dV(\underline{\chi}) \tag{65}$$

Evaluation of these integrals requires evaluation of the individual terms in $\hat{\ddot{\underline{x}}}$:

$$\hat{\ddot{\underline{x}}} = \hat{\ddot{\underline{p}}} + (\hat{\dot{\Omega}} + \hat{\Omega}^2) \cdot (\underline{\chi} + s_i \underline{u}^i(\underline{\chi})) + 2 \hat{\Omega} \cdot \dot{s}_i \underline{u}^i(\underline{\chi}) + \ddot{s}_i \underline{u}^i(\underline{\chi}) \tag{66}$$

The acceleration terms to be determined by the above equations are: $\hat{\ddot{\underline{p}}}$, $\hat{\dot{\Omega}}$, and $\ddot{s}_i$.

Equations 63 and 64 are exactly the equations for translation and rotation that would have been obtained without consideration of geometric nonlinearities. It is in the term $2 s_j \underline{g}^{ij}(\underline{\chi})$ in Equation 65 that the geometric nonlinearity is manifest. Note that all terms of order greater than one in the s_i have been dropped from the expression for acceleration. Linearization at this stage yields different equations for deformation than would have been obtained without consideration of the nonlinear kinematics.

The integrals

$$\int_{vol} 2 s_j \underline{g}^{ij}(\underline{\chi}) \hat{\ddot{\underline{p}}} dV(\underline{\chi}),$$

$$\int_{vol} 2 s_j \underline{g}^{ij}(\underline{\chi}) (\hat{\dot{\Omega}} + \hat{\Omega}^2) \cdot \underline{\chi} dV(\underline{\chi}),$$

and $\int_{vol} 2s_j \underline{g}^{ij}(\underline{\chi})\underline{\hat{F}_X}(\underline{\chi},t)dV(\underline{\chi})$

in Equation 65 are all associated with geometric stiffness matrices[35]. The matrices generated by these integrals differ from the ordinary geometric stiffness matrices in that they are defined with respect to the generalized degrees of freedom s_i rather than with respect to nodal degrees of freedom.

It is shown in a later section how all of the time varying terms in Equations 63,64, and 65 can be factored outside the above volume integrals, in a manner which permits exploitation of a finite element code to perform the integrations over the body.

4 Some Simple Examples

- A massless rotating beam with end mass:

  Returning to the problem of Figure 1, where the reference point at the hub is fixed and the angular velocity at the hub is specified, Equations 63 and 64 are satisfied automatically. The deformations with respect to the local frame are obtained by fixing **R** and $\underline{p}$ (in this case, fixing the hub) and imposing a lateral load equal to $s_1 \kappa L$ at the tip of the beam.

  The resulting displacements define the fields $\underline{u}^1$ and $\underline{g}^{11}$. These fields can be obtained by setting $Y = sL$ in Equation 5 and identifying appropriate coefficients of powers of s. The density in the integrand of Equation 65 is a Dirac delta function at $\underline{\chi} = L$. The relevant terms in the integrand are evaluated at that location and expressed as column vectors:

$$\underline{\chi}_L = \left\{ \begin{array}{c} L \\ 0 \end{array} \right\} \tag{67}$$

$$\underline{u}^1_L = \left\{ \begin{array}{c} 0 \\ L \end{array} \right\} \tag{68}$$

$$\underline{g}^{11}_L = \left\{ \begin{array}{c} -\alpha L \\ 0 \end{array} \right\} \tag{69}$$

$$\underline{\hat{p}} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \tag{70}$$

  The angular velocity tensor is represented by the matrix:

$$\left[\hat{\Omega}\right] = \left[\begin{array}{cc} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{array} \right] \tag{71}$$

  Substitution of these terms into Equation 65, and then linearizing with respect to s_1 yields the anticipated equation (Equation 11) for $\ddot{s}_1$.

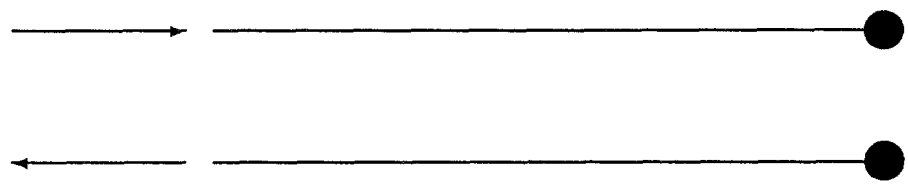

Figure 3. A beam first accelerated linearly to cause axial compression and then accelerated to cause tension.

- A linearly accelerating beam:

  Another interesting example is where the hub in the above problem is prevented from rotating, but is instead caused to move toward the right at a specified acceleration (see Figure 3). Setting

$$\left[\hat{\Omega}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{72}$$

  and

$$\underline{\hat{\ddot{p}}} = \begin{Bmatrix} a \\ 0 \end{Bmatrix}, \tag{73}$$

  substituting these terms and Equations 68 and 69 into Equation 65, and linearizing with respect to s_1 yields the following equation for $\ddot{s}_1$:

$$m\ddot{s}_1 + (\kappa - 2\alpha a)s_1 = 0 \tag{74}$$

  This equation has the interesting ramification that accelerating the mass ahead of the hub will lower the natural frequency while accelerating the hub ahead of the mass will raise the natural frequency. Though these conclusions could also have been obtained through consideration of the level of tension in the beam, the formulation presented here does not require any consideration of that tension at all.

- The beam spin-up maneuver:

  Here we consider a uniform beam accelerated about its hub up to an angular velocity, v_T, well above its first natural cantilevered frequency. The hub angular velocity is imposed as:

$$\dot{\theta}(t) = v_T \frac{2\pi t - P\sin(2\pi t/P)}{2\pi P} \tag{75}$$

Problem Parameters		
Parameter	Symbol	Value
Bending Stiffness	EI	1.4 x 10^4
Beam Length	L	10
Mass/Length	ρ	1.2
Nominal Frequency	$\hat{\omega} = \sqrt{EI/\rho L^4}$	1.1666
Final Angular Velocity	v_T	6.0
Acceleration Period	P	15.0

Table 1. Beam parameters for the problem of a beam accelerated about its hub.

for $t < P$, and

$$\dot{\theta}(t) = v_T \tag{76}$$

for $P < t$.

This is a problem examined by Kane et.al.[13] and by Simo[10][12]. For simplicity, the beam is treated as inextensible. The specifications for the beam and the hub acceleration are presented in Table 4. The parameters of the beam can be combined into a characteristic frequency:

$$\hat{\omega} = \sqrt{EI/\rho L^4} \tag{77}$$

In this problem of planar dynamics, much of the complexity of Equations 63, 64, and 65 simplifies away. Choosing the hub as the reference point results in $\underline{\hat{p}}$ being identically zero. Further, the specification of angular velocity at the hub results in the variation, $\underline{\hat{\lambda}}$, also being identically zero. Equations 63 and 64 are identically satisfied, leaving only Equation 65 to address. Further simplification results from the observation that lateral force fields acting on the beam result in lateral displacements that are proportional to the force field and axial displacements that are quadratic in the force field.

To simplify the numerics of this problem, dimensionless coordinates, forces and displacements are introduced. Position along the beam is ζL where $0 < \zeta < 1$ and L is the length of the beam. The laterally applied force per unit length on the beam is

$$\underline{F_s}(\zeta L) = (EI/L^3) \sum s_k f^k(\zeta) \underline{j} \tag{78}$$

where EI is the bending modulus of the beam, and the resulting displacement is

$$\sum s_k(t) u^k(\zeta) \underline{j} + \sum s_k(t) s_l(t) g^{kl}(\zeta) \underline{i} \tag{79}$$

where $\{f^k(\zeta)\underline{j}\}$ is the basis of dimensionless lateral force fields. The linear displacements satisfy the beam equation:

$$\frac{\partial^4}{\partial \zeta^4} u_k(\zeta) = -f^k(\zeta) \tag{80}$$

Since hub rotation is specified, zero-slope and zero-displacement conditions are imposed at $\zeta = 0$ ($x = 0$) in the above differential equation. Natural boundary conditions hold at $\zeta = 1$ ($x = L$).

The quadratic terms derive from the condition of zero axial strain. The position of particle ζ is represented in the unrotated system as:

$$\underline{z}(\zeta) = (\zeta + s_i s_j g^{ij}(\zeta))\underline{i} + s_k \underline{u}^k \tag{81}$$

The axial strain is

$$\epsilon = \left| \frac{\partial \underline{z}}{\partial \zeta} \right| - 1 \tag{82}$$

and the condition that $\epsilon^2 = 0$ becomes:

$$\frac{\partial}{\partial \zeta} g^{kl}(\zeta) = -\frac{1}{2} \frac{\partial}{\partial \zeta} u_k(\zeta) \frac{\partial}{\partial \zeta} u_l(\zeta) \tag{83}$$

On substitution of these terms into Equation 65, the following system of equations for the coefficients $s_k(t)$: results:

$$\mathcal{M}_{kl} \ddot{s}_l + \left[\hat{\omega}^2 \mathcal{K}_{kl} - \omega^2 (\mathcal{M}_{kl} + 2\mathcal{N}_{kl}) \right] s_l = -\dot{\omega} c_k \tag{84}$$

In the above,

$$\mathcal{M}_{kl} = \int_0^1 u^k(\zeta) u^l(\zeta) d\zeta; \tag{85}$$

$$\mathcal{K}_{kl} = \int_0^1 u^k(\zeta) f^l(\zeta) d\zeta; \tag{86}$$

$$\mathcal{N}_{kl} = \int_0^1 \zeta g^{kl}(\zeta) d\zeta; \tag{87}$$

and

$$c_k = \int_0^1 u^k(\zeta) \zeta d\zeta. \tag{88}$$

In the example below, calculations were performed using a force basis of three polynomials:

$$f^k(\zeta) = \zeta^{k-1} \tag{89}$$

for k=1,2,3.

The resulting lateral displacements are:

$$u^1(\zeta) = \zeta^4/24 - \zeta^3/6 + \zeta^2/4 \tag{90}$$

$$u^2(\zeta) = \zeta^5/120 - \zeta^3/12 + \zeta^2/6 \tag{91}$$

$$u^3(\zeta) = \zeta^6/360 - \zeta^3/18 + \zeta^2/8 \tag{92}$$

The six independent $g^{kl}(\zeta)$ can also be calculated in closed form, but they are too complex to be enlightening, and are not presented here.

When the integrals are performed to evaluate the matrices of Equation 84, the following result is obtained:

$$(\mathcal{M}_{kl}) = \begin{pmatrix} 0.004012 & 0.002914 & 0.002282 \\ 0.002914 & 0.002117 & 0.001658 \\ 0.002282 & 0.001658 & 0.001298 \end{pmatrix} \tag{93}$$

$$(\mathcal{K}_{kl}) = \begin{pmatrix} 0.050000 & 0.036111 & 0.028175 \\ 0.036111 & 0.026190 & 0.020486 \\ 0.028175 & 0.020486 & 0.016049 \end{pmatrix} \tag{94}$$

$$(\mathcal{N}_{kl}) = \begin{pmatrix} -0.002353 & -0.00172 & -0.001351 \\ -0.001720 & -12.57 \cdot 10^{-4} & -9.879 \cdot 10^{-4} \\ -0.001351 & -9.879 \cdot 10^{-4} & -7.764 \cdot 10^{-4} \end{pmatrix} \tag{95}$$

$$(c_k) = \begin{pmatrix} 0.03611 \\ 0.02619 \\ 0.02049 \end{pmatrix} \tag{96}$$

For the purpose of illustration, Equation 84 was solved using a Newmark beta method. The resulting dimensionless tip displacement, as seen in a frame rotating with the hub, is shown in Figure 4. Also shown are the curves resulting when only one basis force ($k = 1$) and two basis forces ($k = 1$ and $k = 2$) are used. The relevant matrices for these lower order approximations are submatrices of the matrices shown above. The three curves are in very good agreement with each other and with the calculations of Simo[10]. Significantly, these calculations were performed using far fewer degrees of freedom than are necessary in other approaches [10, 13].

If one employs as basis force fields the force fields associated with the eigenmodes, even better results are achieved. Figure 4 also shows the dimensionless tip displacement calculated using the force field associated with the first cantilever eigenmode [37]. This plot is especially significant since its calculations involved solving differential equations for only one degree of freedom.

5 Conclusion

Presented above is a method for analyzing general rotating flexible elastic structures. This method permits the use of a reduced number of degrees of freedom while still accommodating the appropriate nonlinearities of the problem. The nonlinear structural

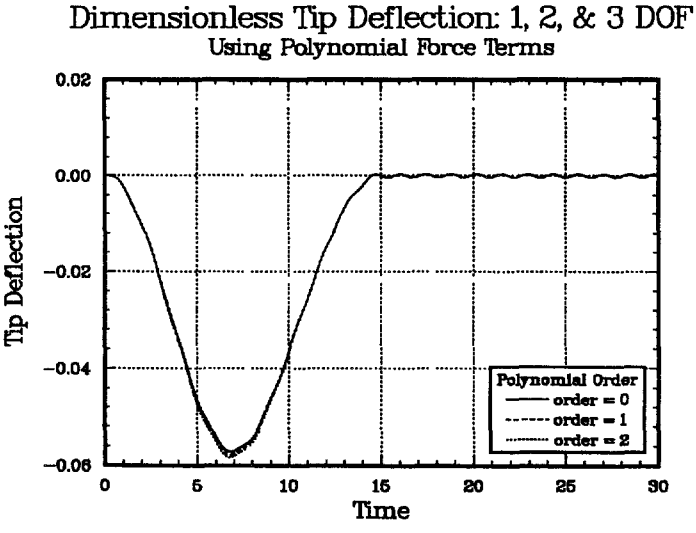

Figure 4. Dimensionless tip displacement as seen in a frame rotating with the hub. Calculations involved force basis sets of one, two, and three polynomial terms.

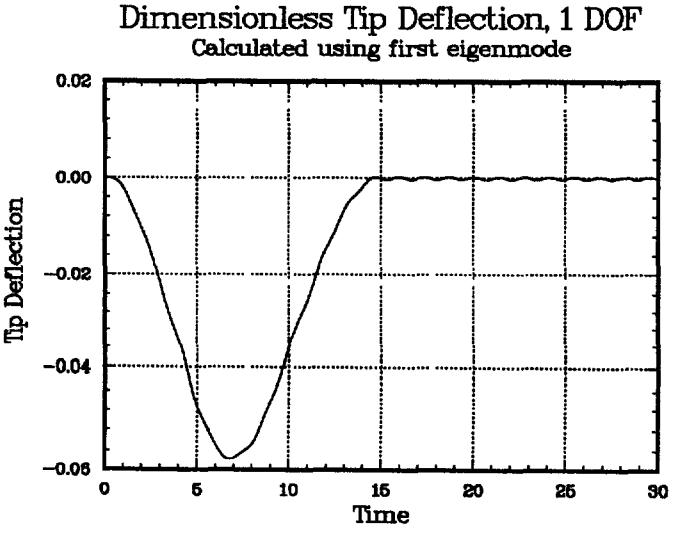

Figure 5. Dimensionless tip displacement as seen in a frame rotating with the hub. Calculations involved a basis of one eigenforce.

analysis required in this method, calculation of the displacement vectors u^i and g^{ij} associated with force fields f^i, occurs prior to the actual dynamics calculations. Nonlinearities that remain in the calculations are no more than those associated with the corresponding rigid body problem.

Though the applications presented in this report were selected to be simple enough to be illuminating, more complex applications include:

- *multi-flexible-body dynamics*

  Though the method presented here is developed for a single, (but *general,*) rotating, flexible structure, this approach can be applied to each of several jointed structures and linked by imposition of appropriate sets of mutual constraints.

- *use of commercial finite element code.*

  The structural calculations of this method can be performed using an existing finite element code in the following manner:

  1. Select basis forces. These basis forces are in general a combination of externally applied loads and eigenforces. The eigenforces are calculated as $f^i = [M]\, d^i$ where d^i is an eigenmode and $[M]$ is the mass matrix. Direct access to the mass matrix is not actually necessary; one temporarily sets all stiffnesses to zero and calculates the reaction forces which result from accelerating all nodes according to d^i.
  2. Evaluate the linear and quadratic modes associated with above basis of forces. These are calculated in the manner in Section 2.
  3. Calculation of necessary volume integrals of Equations 63, 64, and 65 is achieved by calculating all permutations of $[M]\,[\Omega_k]\,u^i$, $[M]\,[\Omega_k]\,g^{ij}$, and $[M]\,[\Omega_k]\,\chi$. In the above, the nine constant tensors $[\Omega_k]$ are a basis for the second order tensors.
  4. Calculate the inner products of Equations 63, 64, and 65. This involves the contraction of vectors each as long as the number of degrees of freedom of the finite element model.
  5. Solve the reduced system of N+6 differential equations in N+6 unknowns where N is number of original basis forces employed. The resulting accelerations, $\ddot{\underline{p}}$, $\dot{\Omega}$, and $\ddot{s}_i$, are used to update the velocity terms, $\dot{\underline{p}}$, Ω, and $\dot{s}_i$ and to update the kinematic and deformation terms, $\underline{p}$, $\mathbf{R}$, and s_i.

  Such an implementation of a finite element code to do the tedious structural calculations prior to the dynamics calculations has been performed. Among calculations performed using this code were those of a flexible orbiting body, in which modal

damping of the flexible degrees of freedom served to increase the coning angle of the overall body. That numerical implementation will be the topic of a future report.

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Appendix A. A Lemma on the integration of $\delta\boldsymbol{\Omega}$

If $\mathbf{T}$ solves the equation

$$\frac{\partial \mathbf{T}}{\partial t} = \boldsymbol{\Pi}(t)\cdot\mathbf{T}(t) \tag{A.1}$$

subject to $\mathbf{T}(0) = \mathbf{I}$ and $\boldsymbol{\Pi}$ being antisymmetric, and if $\delta\boldsymbol{\Psi}$ is defined by

$$\delta\mathbf{T} = \delta\boldsymbol{\Psi}\cdot\mathbf{T} \tag{A.2}$$

then

$$\frac{\partial(\delta\boldsymbol{\Psi})}{\partial t} = \delta\boldsymbol{\Pi} + \boldsymbol{\Pi}\cdot\delta\boldsymbol{\Psi} + \delta\boldsymbol{\Psi}\cdot\boldsymbol{\Pi}^T \tag{A.3}$$

Proof:

Subtracting δ (Equation A.1) from $\frac{\partial}{\partial t}$ (Equation A.2) and post multiplying by $\mathbf{T}^{-1}$, the following is obtained:

$$0 = \frac{\partial(\delta\boldsymbol{\Psi})}{\partial t} + \delta\boldsymbol{\Psi}\cdot\frac{\partial \mathbf{T}}{\partial t}\cdot\mathbf{T}^{-1} - \delta\boldsymbol{\Pi} - \boldsymbol{\Pi}\cdot\delta\mathbf{T}\cdot\mathbf{T}^{-1} \tag{A.4}$$

Using Equations A.1 and A.2 to resolve out $\delta\mathbf{T}$ and $\partial\mathbf{T}/\partial t$ from the above equation results in Equation A.3.

An Application:

Letting $\mathbf{R}(t)$ be the rotation tensor defined by

$$\frac{\partial \mathbf{R}}{\partial t} = \boldsymbol{\Omega}(t)\cdot\mathbf{R}(t) \tag{A.5}$$

and $\mathbf{R}(0) = \mathbf{I}$ where $\boldsymbol{\Omega}(t)$ is the angular velocity tensor at time t, and defining the incremental rotation tensor $\boldsymbol{\Lambda}$ by

$$\delta\mathbf{R} = \delta\boldsymbol{\Lambda}\cdot\mathbf{R} \tag{A.6}$$

the following equation is derived for the time derivative of variation in angular velocity:

$$\delta\boldsymbol{\Omega} = \frac{\partial(\delta\boldsymbol{\Lambda})}{\partial t} - \boldsymbol{\Omega}\cdot\delta\boldsymbol{\Lambda} - \delta\boldsymbol{\Lambda}\cdot\boldsymbol{\Omega}^T \tag{A.7}$$

The above expression is used in the integration by parts of the kinetic energy term in the action integral of Hamilton's principle.

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